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SU(n) bundles over the configuration space of three identical particles moving on $\mathbb{R}^3$

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Abstract. We study the systems of three identical spinless particles moving on $\mathbb{R}^3$ and possessing an SU(n) gauge symmetry. Three such systems are possible, corresponding to the three non-isomorphic SU(n) bundles over $C_3(\mathbb{R}^3)$, the configuration space. We retract $C_3(\mathbb{R}^3)$ to a subcomplex which shows clearly how its homology arises. The three bundles can be realised as pull-backs of the universal bundle $S^7 \rightarrow S^4$ using three non-homotopic maps $C_3(\mathbb{R}^3) \rightarrow S^4$. The two non-trivial bundles admit no flat connection; they do not correspond to Bose, Fermi or parastatistics.

1. Introduction

In the conventional naive quark model, baryons are bound states of three quarks, each with a wavefunction which takes values in the unitary space $\mathbb{C}^3$ on which the group SU(3) acts naturally. The strong interaction between the quarks is assumed to be SU(3) invariant so that the bound states form degenerate multiplets, singlet, octet or decuplet, carrying representations of SU(3).

In gauge theory and in geometric quantisation theory, the state of a physical system whose configuration space is M and whose gauge group is G is represented by a ray in the Hilbert space of L^2 sections of some complex Hermitian vector bundle E over M, associated to a principal bundle $P(M, G)$ over M. The isomorphism class of the principal bundle gives a superselection rule.

Thus the wavefunction of the baryon is no longer a tensor product of three quark wavefunctions each mapping $\mathbb{R}^3$ to $\mathbb{C}^3$, but a section of a vector bundle with structure group SU(3) over the configuration space of three quarks.

In geometric quantisation theory, the identity of identical particles may be given a geometrical significance. Souriau (1970) showed that for gauge group U(1), if one regards a configuration of m identical particles in $\mathbb{R}^3$ as the set $\{r_1, \ldots, r_m\}$ of the m distinct (unordered) positions of the particles, then exactly two principal U(1) bundles exist over the space $C_m(\mathbb{R}^3)$ of these configurations. The trivial bundle corresponds to Bose statistics and the non-trivial bundle to Fermi statistics (see also Bloore 1980). Statistics are thus reduced to topology for the electromagnetic gauge group U(1), and there are no parastatistics.

It is of some interest therefore to investigate what principal G-bundles exist over $C_m(\mathbb{R}^3)$, when G is not U(1) but say U(n) or SU(n), and to interpret these if possible in terms of statistics. We study here a simple case of three identical spin-zero particles moving on $\mathbb{R}^3$. We ignore spin to make it easier. We find there are three principal

SU(n) bundles over $C_3(\mathbb{R}^3)$ and that Bose and Fermi statistics both correspond to the trivial bundle. (Parastatistics requires a U(2)-bundle.) The two non-trivial bundle possess no flat connections; they correspond to a symmetry of the conventional wavefunction $\psi(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$ defined on $\mathbb{R}^9$ which is different from statistics discussed to date. We hope to develop this further in a future paper.

2. The configuration space $C_3(\mathbb{R}^3)$

The configuration space of three distinguishable particles on $\mathbb{R}^3$ is

$$\tilde{C}_r(\mathbb{R}^3) = (\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3)\backslash\Delta$$

where Δ is the set of all configurations in which two or more particles coincide and is excluded in order to make $\tilde{C}_3$ a manifold (Bloore 1980). The manifold $\tilde{C}_3$ is simply connected and has cohomology groups

$$\mathrm{H}^*(\tilde{C}_3, \mathbb{Z}) = \mathbb{Z}, 0, 3\mathbb{Z}, 0, 2\mathbb{Z}$$

calculated by a spectral sequence (Bloore 1980). The configuration space of three indistinguishable particles is

$$C_3(\mathbb{R}^3) = \tilde{C}_3(\mathbb{R}^3)/\mathrm{S}_3$$

where S_3 is the permutation group on three elements. The spectral sequence corresponding to the covering of C_3 by $\tilde{C}_3$ provides the cohomology groups of C_3,

$$\mathrm{H}^*(C_3, \mathbb{Z}) = \mathbb{Z}, 0, Z_2, 0, Z_3; \qquad \mathrm{H}^q(C_3, \mathbb{Z}) = 0 \qquad \text{for } q > 4.$$

The technical detail of the calculation will be presented elsewhere; our purpose here is to give a descriptive account.

If, as in this case, the top cohomology of a manifold M is of degree four then the isomorphism classes of principal SU(n) bundles over M are in (1, 1) correspondence with $[M, S^4] = \mathrm{H}^4(M, \mathbb{Z})$ (Avis and Isham 1979), and their structure group reduces to SU(2) (Isham 1981). Thus there are three non-isomorphic principal SU(n) bundles ξ over C_3, each determined by its second Chern class, $c_2(\xi) \in \mathrm{H}^4(C_3, \mathbb{Z}) = \mathbb{Z}_3$, and we need only treat SU(2) bundles.

It follows from the universal coefficient theorem that the torsion parts of $\mathrm{H}_3(C_3, \mathbb{Z})$ and $\mathrm{H}^4(C_3, \mathbb{Z})$ are equal, so that

$$\mathrm{H}_3(C_3, \mathbb{Z}) = \mathbb{Z}_3; \qquad \mathrm{H}_q(C_3, \mathbb{Z}) = 0 \qquad \text{for } q > 3.$$

A generator D, of $\mathrm{H}_3(C_3, \mathbb{Z})$ is a closed 3-submanifold which does not itself bound a 4-submanifold, W, but for which $3D = \partial W$. It is useful to view C_3 as the space of all triangles in $\mathbb{R}^3$ including colinear ones. We may then realise D as the 3-manifold of equilateral triangles of unit side and fixed centroid, 0 say, and W as the 4-manifold of isosceles triangles of unit base, centroid 0, and height h, $0 \leq h < \sqrt{3}/2$.

As h tends to $\sqrt{3}/2$ the three isosceles triangles lying in the same plane and having bases at 60° to each other all approach the same equilateral triangle and so we need three copies of D to bound W (figure 1).

If a manifold M can be strongly deformation-retracted to a subspace M' then any bundle over M' can be extended to a bundle over M, unique up to isomorphism, of which it is the restriction. M and M' will have the same cohomology. In the next section we give an explicit strong deformation retraction of C_3 to the subcomplex

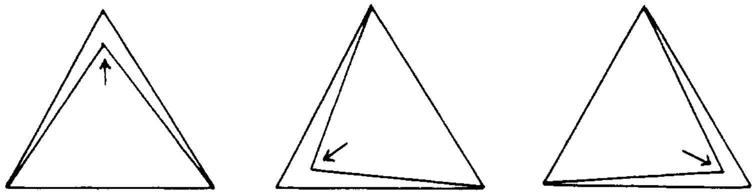

Figure 1. Three isosceles triangles approach the same equilateral triangle.

$W \cup D$ (not actually a manifold), which therefore carries the topology of C_3. In § 4 we will use the fact that the Hopf fibration $S^7 \to S^4$ is 4-universal (it is actually 6-universal (Avis and Isham 1979) for SU(2)), to obtain the three SU(2) bundles over C_3 as pull-backs of $S^7 \to S^4$.

3. The retraction of C_3 to $W \cup D$

The retraction has three steps. Regard C_3 as the nine-dimensional set of all triangles in $\mathbb{R}^3$.

(i) Translate the triangles without rotation until their centroids lie at the origin 0. This retracts C_3 to a six-dimensional space.

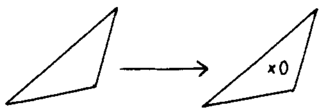

Figure 2. Translation of the triangles to a common centroid O.

(ii) Rescale (dilate) each triangle until its shortest side (or sides) has unit length. This further retracts C_3 to a five-dimensional space.

(iii) Finally, keeping the shortest side fixed in direction and the centroid fixed, move the vertex opposite the shortest side inward towards the midpoint of that side until the triangle becomes isosceles.

In figure 3, the triangles whose shortest side, or whose shortest equal side is parallel to a given direction $\boldsymbol{n}$ are pictured as the triangles ABC where AB is of unit length and parallel to $\boldsymbol{n}$. The points C lie outside or on the boundary of the union of two unit spheres centred on A and B. The two colinear triangles ABE and ABE′ in figure

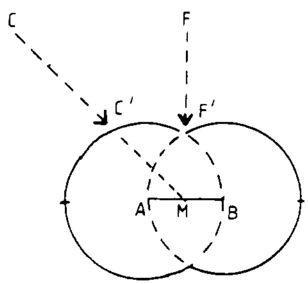

Figure 3. Schematic diagram of step (iii) of the retraction.

3 represent the same triangle with centroid 0. Let M be the midpoint of AB. The retraction takes the scalene triangle ABC to the isosceles triangle ABC′. Colinear triangles retract to colinear triangles, 'tall' isosceles triangles ABF retract to equilateral triangles ABF′. During the retraction the shortest side of the triangle will move sideways, keeping the same direction, since the centroid is held at 0. This is not pictured in figure 3, which depicts the motion of the vertex *relative* to the base.

4. Classification of SU(n) Bundles

In this section we describe the three non-isomorphic SU(2) bundles over $W \cup D$ as pull-backs of the Hopf fibration $\pi: S^7 \to S^4$, which is a 6-universal principal SU(2) bundle. This means that if $\dim M < 6$, every principal SU(2) bundle over M is the pull-back $f^*(S^7)$ of some map $f: M \to S^4$. If f_1 and f_2 are homotopic maps then $f_1^*(S^7)$ and $f_2^*(S^7)$ are isomorphic bundles over M. As noted in § 2, since $W \cup D$ is a strong deformation retract of C_3 we need only consider bundles over $W \cup D$. We now construct three homotopically distinct maps from $W \cup D$ to S^4.

Let I_h be the subset of $W \cup D$ of all isosceles triangles of height h. Then

$$W \cup D = \bigcup_{0 \leq h \leq \sqrt{3}/2} I_h$$

with $I_{\sqrt{3}/2} = D$, the equilateral triangles, and $I_0 \equiv \mathbb{RP}^2$, the colinear triangles. For $0 < h < \sqrt{3}/2$ each I_h is a circle bundle over $\mathbb{RP}^2$. It is not a principal U(1) bundle because it does not carry a continuous U(1) action on the fibres. I_h is also the quotient of the rotation group space $\mathbb{RP}^3$ by the group $\mathbb{Z}_2$ whose generator, γ, acts by rotating the isosceles triangle about its axis of symmetry through an angle of 180°. Hence I_h has a four-fold covering by S^3 and the rotation, γ, regarded as a closed loop in I_h generates the fundamental group $\mathbb{Z}_4$ of I_h. Thus $I_h = S^3/\mathbb{Z}_4$ and by a theorem of Rice (Rice 1969) this space is unique and is the Lens space $L(4, 1)$ with cohomology

$$\mathrm{H}^*(I_h, \mathbb{Z}) = \mathbb{Z}, 0, \mathbb{Z}_4, \mathbb{Z}..$$

In a similar way we may regard S^4 as the suspension of a 3-sphere, i.e.

$$S^4 = \bigcup_{0 \leq h \leq \sqrt{3}/2} S_h^3$$

where $S^3_{\sqrt{3}/2}$ and S^3_0 denote the north and south poles of S^4 and for $0 < h < \sqrt{3}/2 S_h^3$ the 3-sphere of 'latitude' h. Now any continuous map $f: I_h \to S_h^3$, for fixed h, extends to a continuous map $\tilde{f}: W \cup D \to S^4$, sending D to $S^3_{\sqrt{3}/2}$ and I_0 to S_0^3, as follows (figure 4). We have $\bigcup_{0<h<\sqrt{3}/2} I_h = I_{1/2} \times J$ and $\bigcup_{0<h<\sqrt{3}/2} S_h^3 = S^3_{1/2} \times J$ where J is the open interval $(0, \sqrt{3}/2)$. we put $\tilde{f} = f \times \mathrm{id}_J$ and let $\tilde{f}(I_{\sqrt{3}/2}) = S^3_{\sqrt{3}/2}$ and $\tilde{f}(I_0) = S_0^3$. So we study homotopy classes of maps from I_h to S^3.

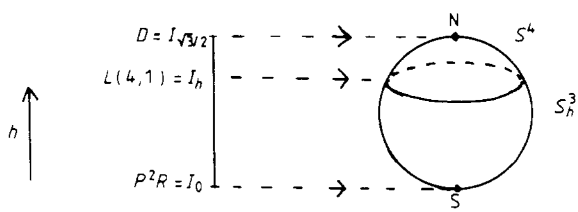

Figure 4. The mappings from $W \cup D$ to S^4.

Since $[I_h, S^3] = H^3(I_h, \mathbb{Z})$ (Avis and Isham 1979), the map f has a homotopy invariant $k \in H^3(I_h, \mathbb{Z}) = \mathbb{Z}$. We show that $\tilde{f}$ is homotopically trivial if and only if $k \bmod 3 = 0$. We split $W \cup D$ into two subsubspaces

$$X = \bigcup_{0 \le h \le 1/2} I_h, \qquad Y = \bigcup_{1/2 \le h \le \sqrt{3}/2} I_h$$

so that $X \cup Y = W \cup D$ and $X \cap Y = I_{1/2}$. In the same way we divide S^4 into northern (N) and southern (S) hemispheres, overlapping only at the equator $S^3_{1/2}$. The map $\tilde{f}$ now induces commutative diagrams involving the Mayer–Vietoris cohomology sequences of both spaces:

$$\begin{array}{ccccccccc}
H^3(S^4) & \longrightarrow & H^3(N)\oplus H^3(S) & \longrightarrow & H^3(S^3) & \longrightarrow & H^4(S^4) & \longrightarrow & H^4(N)\oplus H^4(S)\ldots \\
\downarrow & & \downarrow & & \downarrow f^* & & \downarrow \tilde{f}^* & & \downarrow \\
H^3(W\cup D) & \to & H^3(X)\oplus H^3(Y) & \to & H^3(X\cap Y) & \to & H^4(W\cup D) & \to & H^4(X)\oplus H^4(Y)\ldots
\end{array}$$

where the cohomology groups have $\mathbb{Z}$ coefficients. Since X retracts to $I_0 = \mathbb{RP}^2$, $H^3(X) = H^4(X) = 0$. $H^3(Y)$ is irrelevant, although exactness of the second line (or the fact that Y is retractible to D) implies $H^3(Y) = \mathbb{Z}$. The diagram becomes

$$\begin{array}{ccccc}
 & 0 \to \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} \to 0 \\
 & {\scriptstyle f^* = \times k}\downarrow & & \downarrow {\scriptstyle \tilde{f}^*} \\
0 \to H^3(Y) \to & \mathbb{Z} & \xrightarrow[\pm \bmod 3]{} & \mathbb{Z}_3 \to 0
\end{array}$$

where f^* must be multiplication by some integer k. Even though there is a sign ambiguity in the mod 3 homomorphism, we see from commutativity that $\tilde{f}^*$ is trivial if and only if $k \bmod 3 = 0$. If $k \bmod 3 \neq 0$ then we can obtain representatives of all three homotopy classes of maps $\tilde{f}^*$ by composing f with a general map from S^3 to S^3 of appropriate degree. A suitable map $F: I_{1/2} \to S^3_{1/2}$ is obtained as follows. The Lens space $L(4, 1)$ $(= I_{1/2})$ has a cell decomposition consisting of the ball E^3 surrounded by a 2-sphere S^2 on which certain identifications are made (Hilton and Wylie 1962). S^3 is the ball E^3 whose boundary is all identified to a single point y. Let $f: E^3 \to E^3$ be the identity map and let $f(S^2/\sim) = y$. An argument similar to the one above shows that this f has $k \bmod 3 \neq 0$.

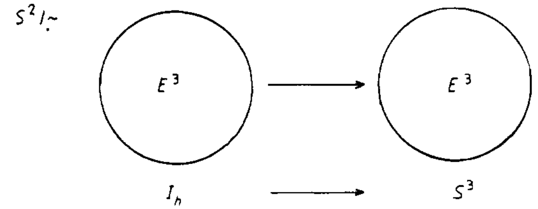

Figure 5. Example of a map $f: I_{1/2} \to S^3_{1/2}$ which is not homotopically trivial.

5. Remarks

Quantum mechanics in the two non-trivial bundles over C_3 has no conventional analogue. Conventionally, a wavefunction of three identical particles is a vector valued function

$$\tilde{\psi} : \tilde{C}_3 \to V$$

taking values in some Hermitian complex vector space V, and is equivariant under permutations,

$$\tilde{\psi}(b\tilde{m}) = D(b)\tilde{\psi}(\tilde{m})$$

where $b \in S_3$, $\tilde{m} \in \tilde{C}_3$ and D is a representation of S_3 on V.

We may regard $\tilde{\psi}$ as a section of the trivial bundle $\tilde{E} = \tilde{C}_3 \times V$, and D as a lift of the action of S_3 on $\tilde{C}_3$ to the bundle $\tilde{E}$. We may then form the vector bundle

$$E = \tilde{E}/S_3 = \tilde{C}_s \times_{S_3} V$$

over C_3, by identifying points in $\tilde{E}$, $(\tilde{m}, \tilde{\psi}(\tilde{m})) \approx (b\tilde{m}, D(b)\tilde{\psi}(\tilde{m}))$. Let $\beta : \tilde{C}_3 \to C_3$ be the covering projection. Then $\tilde{E} = \beta^* E$ and the equivariant sections $\tilde{\psi}$ are the pull-backs $\beta^*\psi$ of sections ψ of E. However, we now show that if E is associated with a principal SU(2) bundle, then that principal bundle is the trivial one. It follows from a theorem of Milnor (Milnor 1957) that a vector bundle over M with standard fibre V admits a flat connection if and only if it is of the form $\tilde{M} \times_{\pi_1 M} V$, where $\tilde{M}$ is the simply connected covering of M and the fundamental group $\pi_1 M$ acts linearly on V. Thus the bundle E admits a flat connection. If E is to be an associated SU(2) bundle the holonomy of this flat connection gives a homomorphism D' from S_3 to SU(2). But only two such homomorphisms exist, the trivial and the alternating one; in both these cases the underlying principal SU(2) bundle $\tilde{C}_3 \times_{D'(S_3)} \mathrm{SU}(2)$ is the trivial one, P_0.

To show that $\tilde{C}_3 \times_{D'(S_3)} \mathrm{SU}(2)$ is trivial when D' is the alternating representation it is enough to specify a globally defined section, or equivalently an antisymmetric map $\alpha : \tilde{C}_3 \to \mathrm{SU}(2)$.

Consider the two antisymmetric maps $\tilde{C}_3 \to \mathbb{R}^3$,

$$\boldsymbol{K}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times (\boldsymbol{r}_3 - \boldsymbol{r}_1) = (\boldsymbol{r}_2 \times \boldsymbol{r}_3) + (\boldsymbol{r}_3 \times \boldsymbol{r}_1) + (\boldsymbol{r}_1 \times \boldsymbol{r}_2)$$

$$\begin{aligned}\boldsymbol{L}(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3) = {} & (\boldsymbol{r}_2 - \boldsymbol{r}_3)[(\boldsymbol{r}_3 - \boldsymbol{r}_1) \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_2)] + (\boldsymbol{r}_3 - \boldsymbol{r}_1)[(\boldsymbol{r}_1 - \boldsymbol{r}_2) \cdot (\boldsymbol{r}_2 - \boldsymbol{r}_3)] \\ & + (\boldsymbol{r}_1 - \boldsymbol{r}_2)[(\boldsymbol{r}_2 - \boldsymbol{r}_3) \cdot (\boldsymbol{r}_3 - \boldsymbol{r}_1)].\end{aligned}$$

The magnitude of $\boldsymbol{K}$ is twice the area of the triangle formed by $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ and its direction is normal to the plane of the triangle in the sense of circulating the triangle from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ to $\boldsymbol{r}_3$. The vector $\boldsymbol{K}$ vanishes only on the colinear triangles.

The vector $\boldsymbol{L}$ lies in the plane of the triangle and vanishes only on equilateral triangles. So $\boldsymbol{K} \cdot \boldsymbol{L} = 0$ and $\boldsymbol{K} + \boldsymbol{L} = \boldsymbol{M}$ say is an antisymmetric vector-valued function of $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_3$ which does not vanish on $\tilde{C}_3$. We may take α to be

$$\alpha = \frac{1}{|\boldsymbol{M}|^2} \begin{pmatrix} M_z & M_x + \mathrm{i}M_y \\ -M_x + \mathrm{i}M_y & M_z \end{pmatrix}.$$

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